

INVARIANTS OF NONCOMMUTATIVE PROJECTIVE SCHEMES

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ABSTRACT. In this note we compute several invariants (e.g. algebraic K -theory, cyclic homology and topological Hochschild homology) of the noncommutative projective schemes associated to Koszul algebras of finite global dimension.

1. INTRODUCTION

Noncommutative projective schemes. Let k be a field and $A = \bigoplus_{n \geq 0} A_n$ a \mathbb{N} -graded Noetherian k -algebra. Throughout the note, we will always assume that A is *connected*, i.e. $A_0 = k$, and *locally finite-dimensional*, i.e. $\dim_k(A_n) < \infty$ for every n . Following Manin [12], Gabriel [6], Artin-Zhang [1], and others, the *noncommutative projective scheme* $\mathrm{qgr}(A)$ associated to A is defined as the quotient category $\mathrm{gr}(A)/\mathrm{tors}(A)$, where $\mathrm{gr}(A)$ stands for the abelian category of finitely generated \mathbb{Z} -graded (right) A -modules and $\mathrm{tors}(A)$ for the Serre subcategory of torsion A -modules. This definition was motivated by Serre's celebrated result [19, Prop. 7.8], which asserts that in the particular case where A is commutative and generated by elements of degree 1 the quotient category $\mathrm{qgr}(A)$ is equivalent to the abelian category of coherent $\mathcal{O}_{\mathrm{Proj}(A)}$ -modules $\mathrm{coh}(\mathrm{Proj}(A))$. For example, when A is the polynomial k -algebra $k[x_1, \dots, x_d]$, with $\deg(x_i) = 1$, we have the following equivalence $\mathrm{qgr}(k[x_1, \dots, x_d]) \simeq \mathrm{coh}(\mathbb{P}^{d-1})$.

Invariants of dg categories. A *dg category* \mathcal{A} is a category enriched over complexes of k -vector spaces; consult Keller's survey [9]. Every (dg) k -algebra B gives naturally rise to a dg category with a single object. Another source of examples is provided by exact categories since the bounded derived category $\mathcal{D}^b(\mathcal{E})$ of every exact category \mathcal{E} admits a canonical dg enhancement $\mathcal{D}_{\mathrm{dg}}^b(\mathcal{E})$; see [9, §4.4]. In what follows, we will denote by $\mathrm{dgc}at(k)$ the category of dg categories and dg functors. A functor $E: \mathrm{dgc}at(k) \rightarrow \mathcal{T}$, with values in a triangulated category, is called:

- (i) *Morita invariant* if it inverts the Morita equivalences; see [9, §4.6].
- (ii) *Localizing* if it sends short exact sequences of dg categories, in the sense of Drinfeld/Keller (see [3][9, §4.6]), to distinguished triangles:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C}) \xrightarrow{\partial} \Sigma E(\mathcal{A}).$$

- (iii) *Co-continuous* if it preserves sequential (homotopy) colimits.

Examples of functors satisfying the conditions (i)-(iii) include nonconnective algebraic K -theory \mathbb{K} , homotopy K -theory KH , étale K -theory K^{et} , the mixed complex C , Hochschild homology HH , cyclic homology HC , and topological Hochschild

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homology THH ; see [22, §8.2]. Some other functors such as periodic cyclic homology HP and negative cyclic homology HN only satisfy conditions (i)-(ii). When applied to B , resp. to $\mathcal{D}_{\text{dg}}^b(\mathcal{E})$, all the preceding invariants of dg categories reduce to the corresponding invariants of the (dg) k -algebra B , resp. of the exact category \mathcal{E} .

Notation 1.1. Given a functor $E: \text{dgc}at(k) \rightarrow \mathcal{T}$, an object $o \in \mathcal{T}$, an integer $m \in \mathbb{Z}$, and a dg category \mathcal{A} , let us write $E_m^o(\mathcal{A}) := \text{Hom}_{\mathcal{T}}(\Sigma^m(o), E(\mathcal{A}))$. Whenever \mathcal{T} is symmetric monoidal with \otimes -unit $\mathbf{1}$, we will write $E_m(\mathcal{A})$ instead of $E_m^1(\mathcal{A})$.

Statement of results. Let k be a field and $A = \bigoplus_{n \geq 0} A_n$ a \mathbb{N} -graded Noetherian k -algebra. Assume that A is Koszul and has finite global dimension d . Under these assumptions, the Hilbert series $h_A(t) := \sum_{n \geq 0} \dim_k(A_n)t^n \in \mathbb{Z}[[t]]$ is invertible and its inverse $h_A(t)^{-1}$ is a polynomial $1 - \beta_1 t + \beta_2 t^2 - \dots + (-1)^d \beta_d t^d$ of degree d , where β_i stands for the dimension of the k -vector space $\text{Tor}_i^A(k, k)$ (or $\text{Ext}_A^i(k, k)$). In what follows, we write $\beta := \beta_d$. Our main result is the following computation:

Theorem 1.2. *Let A be a k -algebra as above and $E: \text{dgc}at(k) \rightarrow \mathcal{T}$ a functor satisfying conditions (i)-(iii). Assume that \mathcal{T} is R -linear for a commutative ring R .*

(i) *For every compact object $o \in \mathcal{T}$, we have R -module isomorphisms*

$$(1.3) \quad E_m(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \simeq R[t]/\langle h'_A(t)^{-1} \rangle \otimes_R E_m^o(k) \quad m \in \mathbb{Z},$$

where $h'_A(t)^{-1} = 1 - \beta'_1 t + \beta'_2 t^2 - \dots + (-1)^{d'} \beta'_{d'} t^{d'}$ stands for the image of the polynomial $h_A(t)^{-1}$ in $R[t]$. In what follows, we write $\beta' := \beta'_{d'}$.

(ii) *Assume moreover that $1/\beta' \in R$ and that \mathcal{T} is compactly generated. Under these assumptions, we have an isomorphism $E(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \simeq E(k)^{\oplus d'}$.*

Remark 1.4. (i) If $\beta = 1$, then $\beta' = \beta$ and $d' = d$. As proved in [21, Cor. 0.2], in the particular case where $d = 3$, we always have $h_A(t)^{-1} = (1 - t)^3$.

(ii) If R is a field, then $1/\beta' \in R$. Moreover, $\beta' = \beta$ and $d' = d$ if and only if the characteristic of R does not divide β .

Corollary 1.5. *Let A be a k -algebra as above and $E: \text{dgc}at(k) \rightarrow \mathcal{T}$ a functor satisfying conditions (i)-(iii). Assume moreover that \mathcal{T} is compactly generated. Under these assumptions, we have an isomorphism $E(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A)))_{1/\beta'} \simeq E(k)_{1/\beta'}^{\oplus d'}$ in the $\mathbb{Z}[1/\beta']$ -linearized triangulated category¹ $\mathcal{T}_{1/\beta'}$.*

Proof. By construction, the triangulated category $\mathcal{T}_{1/\beta'}$ is compactly generated and $R[1/\beta']$ -linear. Moreover, the $\mathbb{Z}[1/\beta']$ -linearization functor $(-)_1/\beta': \mathcal{T} \rightarrow \mathcal{T}_{1/\beta'}$ is triangulated and preserves arbitrary direct sums. Therefore, the proof follows from Theorem 1.2(ii) applied to $E = E(-)_{1/\beta'}$ (with $R = R[1/\beta']$). \square

Example 1.6 (Algebraic K -theory). Nonconnective algebraic K -theory gives rise to a functor $\mathbb{K}: \text{dgc}at(k) \rightarrow \text{Ho}(\text{Spt})$, with values in the homotopy category of spectra, satisfying conditions (i)-(iii); see [22, §8.2.1]. Therefore, by applying Theorem 1.2(i) to $E = \mathbb{K}$ (with $R = \mathbb{Z}$) and to the sphere spectrum $o = \mathbb{S}$, we obtain isomorphisms²

$$(1.7) \quad \mathbb{K}_m(\text{qgr}(A)) \simeq \mathbb{Z}[t]/\langle h_A(t)^{-1} \rangle \otimes_{\mathbb{Z}} \mathbb{K}_m(k) \quad m \in \mathbb{Z}.$$

¹Let \mathcal{G} be a set of compact generators of \mathcal{T} . Recall that $\mathcal{T}_{1/\beta'}$ may be defined as the Verdier quotient of \mathcal{T} by the smallest localizing (=closed under arbitrary direct sums) triangulated subcategory containing the objects $\{\text{cone}(\beta \cdot \text{id}_o) \mid o \in \mathcal{G}\}$.

²In the particular case where $m = 0$, the isomorphism (1.7) was originally established by Mori-Smith in [14, Thm. 2.3].

Moreover, since the triangulated category $\mathrm{Ho}(\mathrm{Spt})$ is compactly generated, Corollary 1.5 implies that $\mathbb{K}(\mathrm{qgr}(A))_{1/\beta} \simeq \mathbb{K}(k)_{1/\beta}^{\oplus d}$. All the above holds *mutatis mutandis* with \mathbb{K} replaced by KH or K^{et} .

Example 1.8 (Mixed complex). Following Kassel [8], a *mixed complex* is a (right) dg module over the k -algebra of dual numbers $\Lambda := k[\epsilon]/\epsilon^2$ with $\deg(\epsilon) = -1$ and $d(\epsilon) = 0$. The mixed complex gives rise to a functor $C: \mathrm{dgc}at(k) \rightarrow \mathcal{D}(\Lambda)$, with values in the derived category of Λ , satisfying conditions (i)-(iii); see [22, §8.2.4]. Therefore, since the category $\mathcal{D}(\Lambda)$ is compactly generated, by applying Theorem 1.2(ii) to $E = C$ (with $R = k$), we obtain an isomorphism $C(\mathrm{qgr}(A)) \simeq C(k)^{\oplus d'}$.

Example 1.9 (Cyclic homology and its variants). As explained by Keller in [11, §2.2], Hochschild homology HH , cyclic homology HC , periodic cyclic homology HP , and negative cyclic homology HN , can be recovered from the mixed complex C . Therefore, making use of Example 1.8, we conclude that

$$\begin{aligned} HH(\mathrm{qgr}(A)) &\simeq HH(k)^{\oplus d'} & HC(\mathrm{qgr}(A)) &\simeq HC(k)^{\oplus d'} \\ HP(\mathrm{qgr}(A)) &\simeq HP(k)^{\oplus d'} & HN(\mathrm{qgr}(A)) &\simeq HN(k)^{\oplus d'}. \end{aligned}$$

Example 1.10 (Topological Hochschild homology). Topological Hochschild homology gives rise to a (lax symmetric monoidal) functor $THH: \mathrm{dgc}at(k) \rightarrow \mathrm{Ho}(\mathrm{Spt})$ satisfying conditions (i)-(iii); see [22, §8.2.8]. Since the “inclusion of the 0th skeleton” yields a ring homomorphism $k \rightarrow THH_0(k)$, the abelian groups THH_* are then naturally equipped with a k -linear structure. Therefore, using the fact that the triangulated category $\mathrm{Ho}(\mathrm{Spt})$ is (compactly) generated by the sphere spectrum \mathbb{S} , an argument similar to the one used in the proof of Theorem 1.2(ii) allows us to conclude that $THH(\mathrm{qgr}(A)) \simeq THH(k)^{\oplus d'}$. For example, in the particular where $k = \mathbb{F}_p$, with p a prime number, we have the following isomorphisms:

$$THH_m(\mathrm{qgr}(A)) \simeq \begin{cases} (\mathbb{F}_p)^{\oplus d'} & m \geq 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively speaking, Theorem 1.2 (as well as Corollary 1.5 and Examples 1.6-1.10) shows that all the different invariants of a noncommutative projective scheme $\mathrm{qgr}(A)$ are completely determined by the Hilbert series $h_A(t)$ of A .

Theorem 1.2 (as well as Corollary 1.5) may be applied to the following algebras:

Example 1.11 (Quantum polynomial algebras). Choose constant elements $q_{ij} \in k^\times$ with $1 \leq i < j \leq d$. The following \mathbb{N} -graded Noetherian k -algebra

$$A := k\langle x_1, \dots, x_d \rangle / \langle x_j x_i - q_{ij} x_i x_j \mid 1 \leq i < j \leq d \rangle,$$

with $\deg(x_i) = 1$, is called the *quantum polynomial algebra* associated to q_{ij} . This algebra is Koszul, has global dimension d , and $h_A(t)^{-1} = (1-t)^d$; see [13, §1].

Example 1.12 (Quantum matrix algebras). Choose a $q \in k^\times$. The \mathbb{N} -graded Noetherian k -algebra A defined as the quotient of $k\langle x_1, x_2, x_3, x_4 \rangle$ by the relations

$$\begin{aligned} x_1 x_2 &= q x_2 x_1 & x_1 x_3 &= q x_3 x_1 & x_1 x_4 - x_4 x_1 &= (q - q^{-1}) x_2 x_3 \\ x_2 x_3 &= x_3 x_2 & x_2 x_4 &= q x_4 x_2 & x_1 x_4 &= q x_4 x_3, \end{aligned}$$

with $\deg(x_i) = 1$, is called the *quantum matrix algebra* associated to q . This algebra is Koszul, has global dimension 4, and $h_A(t)^{-1} = (1-t)^4$; see [13, §1].

Example 1.13 (Sklyanin algebras). Let C be a smooth elliptic curve, $\sigma \in \text{Aut}(C)$ an automorphism given by translation under the group law, and \mathcal{L} a line bundle on C of degree $d \geq 3$. We write $\Gamma_\sigma \subset C \times C$ for the graph of σ and V for the d -dimensional k -vector space $H^0(C, \mathcal{L})$. The \mathbb{N} -graded Noetherian k -algebra $A := T(V)/R$, where

$$R := H^0(C \times C, (\mathcal{L} \boxtimes \mathcal{L})(-\Gamma_\sigma)) \subset H^0(C \times C, \mathcal{L} \boxtimes \mathcal{L}) = V \otimes V,$$

is called the *Sklyanin algebra* associated to the triple (C, σ, \mathcal{L}) . This algebra is Koszul, has global dimension d , and $h_A(t)^{-1} = (1-t)^d$; see [4][24, §1].

Example 1.14 (Homogenized enveloping algebras). Let \mathfrak{g} be a finite dimensional Lie algebra. The following \mathbb{N} -graded Noetherian k -algebra (z is a new variable)

$$A := T(\mathfrak{g} \oplus kz) / \langle \{z \otimes x - x \otimes z \mid x \in \mathfrak{g}\} \cup \{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathfrak{g}\} \rangle,$$

is called the *homogenized enveloping algebra* of \mathfrak{g} . This algebra is Koszul, has global dimension $d := \dim(\mathfrak{g}) + 1$, and $h_A(t)^{-1} = (1-t)^d$; see [20, §12].

Example 1.15. Let k be an uncountable algebraically closed field. Choose a pair of elements (θ, ρ) of k^\times which are algebraically independent over the prime field of k and write $\Theta := \frac{\theta-1}{\theta+1}$ and $\Delta := \frac{\rho-1}{\rho+1}$. Under these assumptions and notations, the \mathbb{N} -graded Noetherian k -algebra $A := k\langle x_1, x_2, x_3, x_4 \rangle / \langle f_1, \dots, f_6 \rangle$, where

$$\begin{aligned} f_1 &:= x_1(\Theta x_1 - x_3) + x_3(x_1 - \Theta x_3) & f_2 &:= x_1(\Theta x_2 - x_4) + x_3(x_2 - \Theta x_4) \\ f_3 &:= x_2(\Theta x_1 - x_3) + x_4(x_1 - \Theta x_3) & f_4 &:= x_2(\Theta x_2 - x_4) + x_4(x_2 - \Theta x_4) \\ f_5 &:= x_1(\Delta x_1 - x_2) + x_4(x_1 - \Delta x_2) & f_6 &:= x_1(\Delta x_3 - x_4) + x_4(x_3 - \Delta x_4), \end{aligned}$$

is Koszul, has global dimension 4, and $h_A(t)^{-1} = (1-t)^4$; see [18, Thm. 3.5].

Gorenstein algebras. Recall that a \mathbb{N} -graded Noetherian k -algebra $A = \bigoplus_{n \geq 0} A_n$ is called *Gorenstein*, with Gorenstein parameter l , if it has finite injective dimension m and $\mathbf{R}\text{Hom}_A(k, A) \simeq \Sigma^{-m}k(l)$, where $k(l)$ stands for the \mathbb{Z} -graded (right) A -module $k(l)_n := k_{n+l}$. Let us assume moreover that A has finite global dimension d ; this implies that $d = m$. Under these assumptions, a remarkable result of Orlov (see [15, Cor. 2.7]) asserts that the bounded derived category $\mathcal{D}^b(\text{qgr}(A))$ admits a full exceptional collection of length l . This leads naturally to the following result:

Theorem 1.16. *Let A be a \mathbb{N} -graded Noetherian k -algebra and E a functor satisfying conditions³ (i)-(ii). Assume that A is Gorenstein, with Gorenstein parameter l , and has finite global dimension d . Under these assumptions, we have an isomorphism $E(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \simeq E(k)^{\oplus l}$.*

Proof. As explained in [22, §2.4.2 and §8.4.5], every functor E satisfying conditions (i)-(ii) sends a full exceptional collections of length l to the direct sum $E(k)^{\oplus l}$. \square

Remark 1.17. (i) Since A is connected and has finite global dimension, the Hilbert series $h_A(t)$ is invertible and its inverse $h_A(t)^{-1}$ is a polynomial. Moreover, the Gorenstein condition implies that $h_A(t)^{-1}$ is monic and has degree l .
(ii) As proved in [16, Chap. 2 Thm. 2.5], A is moreover Koszul if and only if $d = l$.

Note that Theorem 1.2 does *not* follow from Theorem 1.16 because, in general, Koszulness does *not* imply⁴ Gorensteiness. For instance, the algebras A of Example 1.15 are Koszul but *not* Gorenstein; see [18, Thm. 3.5]. In this latter example,

³More generally, condition (ii) can be replaced by *additivity* in the sense of [22, Def. 2.1].

⁴In the particular case where $d = 3$, Koszulness indeed implies Gorensteiness; see [21, Cor. 0.2].

we have moreover $\dim_k(\text{Ext}_A^i(k, A)) = \infty$ for $i = 2, 3, 4$; see [18, Prop. 5.11]. Consequently, the k -linear triangulated categories $\mathcal{D}^b(\text{qgr}(A))$ are *not* even Ext-finite.

2. PROOF OF THEOREM 1.2

Recall from Quillen [17, §2] that an exact category \mathcal{E} is an additive category equipped with a family of short exact sequences satisfying some standard conditions. In order to simplify the exposition, given an exact functor $F: \mathcal{E} \rightarrow \mathcal{E}'$, we will still denote by $F: \mathcal{D}_{\text{dg}}^b(\mathcal{E}) \rightarrow \mathcal{D}_{\text{dg}}^b(\mathcal{E}')$ the induced dg functor. We start with the following general result of independent interest:

Proposition 2.1. *Let $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ be a short exact sequence of exact functors $F_1, F_2, F_3: \mathcal{E} \rightarrow \mathcal{E}'$. Given any localizing functor $E: \text{dgc}(\mathcal{E}) \rightarrow \mathcal{T}$, we have the following equality $E(F_2) = E(F_1) + E(F_3)$.*

Proof. Let $\text{Ex}(\mathcal{E}')$ be the category of short exact sequences $\varepsilon = (a \rightarrow b \rightarrow c)$ in \mathcal{E}' ; this is also an exact category with short exact sequence defined componentwise. By construction, $\text{Ex}(\mathcal{E}')$ comes equipped with the following exact functors

$$\iota_1: \mathcal{E}' \longrightarrow \text{Ex}(\mathcal{E}') \quad a \mapsto (a = a \rightarrow 0) \quad \iota_2: \mathcal{E}' \longrightarrow \text{Ex}(\mathcal{E}') \quad a \mapsto (0 \rightarrow a = a)$$

$$\pi_1: \text{Ex}(\mathcal{E}') \xrightarrow{\varepsilon \mapsto a} \mathcal{E}' \quad \pi_2: \text{Ex}(\mathcal{E}') \xrightarrow{\varepsilon \mapsto b} \mathcal{E}' \quad \pi_3: \text{Ex}(\mathcal{E}') \xrightarrow{\varepsilon \mapsto c} \mathcal{E}'$$

satisfying the equalities $\pi_1 \circ \iota_1 = \pi_2 \circ \iota_1 = \text{id}$, $\pi_3 \circ \iota_1 = \pi_1 \circ \iota_2 = 0$, and $\pi_2 \circ \iota_2 = \pi_3 \circ \iota_2 = \text{id}$. Moreover, we have the following short exact sequence of dg categories

$$0 \longrightarrow \mathcal{D}_{\text{dg}}^b(\mathcal{E}') \xrightarrow{\iota_1} \mathcal{D}_{\text{dg}}^b(\text{Ex}(\mathcal{E}')) \xrightarrow{\pi_3} \mathcal{D}_{\text{dg}}^b(\mathcal{E}') \longrightarrow 0$$

and consequently the following distinguished triangle

$$E(\mathcal{D}_{\text{dg}}^b(\mathcal{E}')) \xrightarrow{E(\iota_1)} E(\mathcal{D}_{\text{dg}}^b(\text{Ex}(\mathcal{E}'))) \xrightarrow{E(\pi_3)} E(\mathcal{D}_{\text{dg}}^b(\mathcal{E}')) \xrightarrow{\partial} \Sigma E(\mathcal{D}_{\text{dg}}^b(\mathcal{E}')).$$

Since $\pi_3 \circ \iota_2 = \text{id}$, the preceding triangle splits and induces an isomorphism

$$(2.2) \quad [E(\iota_1) \ E(\iota_2)]: E(\mathcal{D}_{\text{dg}}^b(\mathcal{E}')) \oplus E(\mathcal{D}_{\text{dg}}^b(\mathcal{E}')) \xrightarrow{\sim} E(\mathcal{D}_{\text{dg}}^b(\text{Ex}(\mathcal{E}'))).$$

Note that a short exact sequence of exact functors $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ is the same data as an exact functor $F: \mathcal{E} \rightarrow \text{Ex}(\mathcal{E}')$. Therefore, by combining the equalities $E(\pi_2) \circ [E(\iota_1) \ E(\iota_2)] = [\text{id} \ \text{id}]$ and $(E(\pi_1) + E(\pi_3)) \circ [E(\iota_1) \ E(\iota_2)] = [\text{id} \ \text{id}]$ with the fact that (2.2) is an isomorphism, we conclude that $E(\pi_2) = E(\pi_1) + E(\pi_3)$. The proof follows now from the equalities $\pi_1 \circ F = F_1$, $\pi_2 \circ F = F_2$, and $\pi_3 \circ F = F_3$. \square

Let $B = \bigoplus_{n \geq 0} B_n$ be a \mathbb{N} -graded k -algebra and $\text{grproj}(B)$ the exact category of finitely generated projective \mathbb{Z} -graded (right) B -modules. The following general computation is also of independent interest:

Proposition 2.3. *We have an isomorphism $E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(B))) \simeq \bigoplus_{-\infty}^{+\infty} E(B_0)$.*

Proof. Consider B_0 as an \mathbb{N} -graded k -algebra concentrated in degree zero. The canonical inclusion $B_0 \rightarrow B$ and projection $B \rightarrow B_0$ of \mathbb{N} -graded k -algebras give rise to the following base-change exact functors:

$$\begin{aligned} \varphi: \text{grproj}(B_0) &\longrightarrow \text{grproj}(B) & P &\mapsto P \otimes_{B_0} B \\ \psi: \text{grproj}(B) &\longrightarrow \text{grproj}(B_0) & P &\mapsto P \otimes_B B_0. \end{aligned}$$

Since $\psi \circ \varphi = \text{id}$, it follows from Lemma 2.5 below that φ and ψ give rise to inverse isomorphisms between $E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(B)))$ and $E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(B_0)))$.

Now, note that we have the following canonical equivalence of exact categories

$$(2.4) \quad \text{grproj}(B_0) \xrightarrow{\simeq} \coprod_{n \in \mathbb{Z}} \text{proj}(B_0) \quad P \mapsto \{P_n\}_{n \in \mathbb{Z}},$$

where $\text{proj}(B_0)$ stands for the exact category of finitely generated projective (right) B_0 -modules. Since the dg category $\mathcal{D}_{\text{dg}}^b(\text{proj}(B_0))$ is Morita equivalent to the k -algebra B_0 and the functor E is co-continuous, we then conclude from the equivalence (2.4) that $E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(B_0))) \simeq \oplus_{-\infty}^{+\infty} E(B_0)$. This finishes the proof. \square

Lemma 2.5. *The following endomorphism is equal to the identity*

$$E(\varphi \circ \psi): E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(B))) \longrightarrow E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(B))).$$

Proof. Let $P \in \text{grproj}(B)$. Note first that the exact endofunctor $\varphi \circ \psi$ of $\text{grproj}(B)$ is given by $P \mapsto \bigoplus_{n \in \mathbb{Z}} \psi(P)_n \otimes_{B_0} B(-n)$. Since the functor E is co-continuous, this yields the following equality

$$(2.6) \quad E(\varphi \circ \psi) = \sum_{n \in \mathbb{Z}} E(\psi(-)_n \otimes_{B_0} B(-n)).$$

Given a finitely generated projective \mathbb{Z} -graded (right) B -module P and an integer $m \in \mathbb{Z}$, let us write $F_m(P)$ for the \mathbb{Z} -graded (right) B -submodule of P generated by the elements $\bigcup_{n \leq m} P_n$. In the same vein, given an integer $q \geq 0$, let us denote by $\text{grproj}_q(B)$ the full subcategory of $\text{grproj}(B)$ consisting of those \mathbb{Z} -graded (right) B -module P such that $F_{-(q+1)}(P) = 0$ and $F_q(P) = P$. Note that by definition we have an exhaustive increasing filtration

$$(2.7) \quad \text{grproj}_0(B) \subset \text{grproj}_1(B) \subset \cdots \subset \text{grproj}_q(B) \subset \cdots \subset \text{grproj}(B).$$

As explained by Quillen in [17, pages 99-100], for every $m \in \mathbb{Z}$, the assignment $P \mapsto F_m(P)/F_{m-1}(P)$ is an exact endofunctor of $\text{grproj}(B)$. Moreover, we have a canonical isomorphism of exact functors between $\psi(-)_m \otimes_{B_0} B(-m)$ and $F_m(-)/F_{m-1}(-)$. Consequently, we obtain the following equality

$$(2.8) \quad \sum_{n \in \mathbb{Z}} E(\psi(-)_n \otimes_{B_0} B(-n)) = \sum_{n \in \mathbb{Z}} E(F_n(-)/F_{n-1}(-)).$$

Now, note that every \mathbb{Z} -graded (right) B -module $P \in \text{grproj}_q(B)$ admits a canonical filtration $0 = F_{-(q+1)}(P) \subset \cdots \subset F_q(P) = P$. This yields a sequence $0 = F_{-(q+1)}(-) \rightarrow \cdots \rightarrow F_q(-) = \text{id}$ of exact endofunctors of $\text{grproj}_q(B)$. Consequently, an inductive argument using the above general Proposition 2.1 implies that the sum $\sum_{n=-q}^q E(F_n(-)/F_{n-1}(-))$ is equal to the identity of $E(\mathcal{D}_{\text{dg}}^b(\text{grproj}_q(B)))$. Finally, using the fact that the above filtration (2.7) of $\text{grproj}(B)$ is exhaustive and that the functor E is co-continuous, we hence conclude that

$$(2.9) \quad \sum_{n \in \mathbb{Z}} E(F_n(-)/F_{n-1}(-)) = \text{id}.$$

The proof follows now from the combination of (2.6) with (2.8)-(2.9). \square

Recall that A is a (connected and locally finite-dimensional) \mathbb{N} -graded Noetherian k -algebra, which we assume to be Koszul and of finite global dimension d .

Proposition 2.10. *We have a short exact sequence of dg categories*

$$(2.11) \quad 0 \longrightarrow \mathcal{D}_{\text{dg}}^b(\text{tors}(A)) \longrightarrow \mathcal{D}_{\text{dg}}^b(\text{gr}(A)) \longrightarrow \mathcal{D}_{\text{dg}}^b(\text{qgr}(A)) \longrightarrow 0.$$

Proof. As explained by Keller in [9, Thm. 4.11], (2.11) is a short exact sequence of dg categories if and only if the associated sequence of triangulated categories

$$(2.12) \quad \mathcal{D}^b(\text{tors}(A)) \longrightarrow \mathcal{D}^b(\text{gr}(A)) \longrightarrow \mathcal{D}^b(\text{qgr}(A))$$

is exact sequence in the sense of Verdier. By definition, we have a short exact sequence of abelian categories $0 \rightarrow \text{tors}(A) \rightarrow \text{gr}(A) \rightarrow \text{qgr}(A) \rightarrow 0$. Therefore, thanks to [10, Lem. 1.15] (consult also [7]), in order to show that (2.12) is exact in the sense of Verdier, it suffices to prove the following condition: given a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ in the abelian category $\text{gr}(A)$, with $L \in \text{tors}(A)$, there exists a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

with L' and L'' belonging to $\text{tors}(A)$. Recall that the category $\text{tors}(A)$ of torsion A -modules is defined as the full subcategory of $\text{gr}(A)$ consisting of those \mathbb{Z} -graded (right) A -modules which are (globally) finite-dimensional over k . Given a \mathbb{Z} -graded (right) A -module M and an integer $m \in \mathbb{Z}$, let us write $M_{\geq m}$ for the (right) A -submodule $\bigoplus_{n \geq m} M_n$ of M . Since by assumption L is torsion and M is finitely generated, there exists an integer $m \gg 0$ such that $L \cap M_{\geq m} = 0$. Consequently, we can construct the following morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M/M_{\geq m} & \longrightarrow & M/\langle M_{\geq m} + L \rangle & \longrightarrow & 0. \end{array}$$

The proof follows now from the fact that, by construction, the \mathbb{Z} -graded (right) A -modules $M/M_{\geq m}$ and $M/\langle M_{\geq m} + L \rangle$ belong to $\text{tors}(A)$. \square

Remark 2.13. By assumption, the functor E is localizing. Therefore, the short exact sequence of dg categories (2.11) gives rise to a distinguished triangle:

$$E(\mathcal{D}_{\text{dg}}^b(\text{tors}(A))) \longrightarrow E(\mathcal{D}_{\text{dg}}^b(\text{gr}(A))) \longrightarrow E(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \xrightarrow{\partial} \Sigma E(\mathcal{D}_{\text{dg}}^b(\text{tors}(A))).$$

Since A has finite global dimension, the inclusion of categories $\text{grproj}(A) \subset \text{gr}(A)$ induces a Morita equivalence $\mathcal{D}_{\text{dg}}^b(\text{grproj}(A)) \rightarrow \mathcal{D}_{\text{dg}}^b(\text{gr}(A))$. Therefore, by first using the general Proposition 2.3 (with $B = A$) and then by applying the functor E to the preceding Morita equivalence, we obtain an induced isomorphism

$$(2.14) \quad \bigoplus_{-\infty}^{+\infty} E(k) \simeq E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(A))) \xrightarrow{\simeq} E(\mathcal{D}_{\text{dg}}^b(\text{gr}(A))).$$

Proposition 2.15. *We have a Morita equivalence*

$$(2.16) \quad \mathcal{D}_{\text{dg}}^b(\text{tors}(A)) \longrightarrow \mathcal{D}_{\text{dg}}^b(\text{grproj}(A^!)),$$

where $A^!$ stands for the Koszul dual k -algebra of A .

Proof. Given a \mathbb{N} -graded k -algebra $B = \bigoplus_{n \geq 0} B_n$, let us denote by $\text{Gr}(B)$ the category of all \mathbb{Z} -graded (right) B -modules and by $\mathcal{D}(\text{Gr}(B))$ the associated (unbounded) derived category. Following Beilinson-Ginzburg-Soergel [2, §2.12], let $\mathcal{D}^\downarrow(\text{Gr}(B))$, resp. $\mathcal{D}^\uparrow(\text{Gr}(B))$, be the full subcategory of $\mathcal{D}(\text{Gr}(B))$ consisting of

those cochain complexes of \mathbb{Z} -graded (right) B -modules M such that for some integer $m \gg 0$ we have $M_n^q \neq 0 \Rightarrow (q \geq -m \text{ or } q + n \leq m)$, resp. $M_n^q \neq 0 \Rightarrow (q \leq -m \text{ or } q + n \geq -m)$. These categories admit canonical dg enhancements $\mathcal{D}_{\text{dg}}(\text{Gr}(B))$, $\mathcal{D}_{\text{dg}}^\downarrow(\text{Gr}(B))$, and $\mathcal{D}_{\text{dg}}^\uparrow(\text{Gr}(B))$. Now, recall from [2, Thm. 2.12.1] (consult also [5, §2]) the construction of the Koszul duality dg functor $\mathcal{D}_{\text{dg}}(\text{Gr}(A)) \rightarrow \mathcal{D}_{\text{dg}}(\text{Gr}(A^!))$. As proved in *loc. cit.*, this dg functor restricts to a Morita equivalence

$$(2.17) \quad \mathcal{D}_{\text{dg}}^\downarrow(\text{Gr}(A)) \longrightarrow \mathcal{D}_{\text{dg}}^\uparrow(\text{Gr}(A^!))$$

which sends the \mathbb{Z} -graded (right) A -modules $k(i), i \in \mathbb{Z}$, to the \mathbb{Z} -graded (right) $A^!$ -modules $\Sigma^{-i}A^!(i), i \in \mathbb{Z}$. Therefore, making use of the general Lemma 2.18 below (with $B = A$ and $B = A^!$), we conclude that (2.17) restricts furthermore to the above Morita equivalence (2.16). \square

Lemma 2.18. *Let $B = \bigoplus_{n \geq 0} B_n$ be a (connected and locally finite-dimensional) \mathbb{N} -graded Noetherian k -algebra. The smallest thick triangulated subcategory of $\mathcal{D}^b(\text{gr}(B))$ containing the \mathbb{Z} -graded (right) B -modules $\{k(i) \mid i \in \mathbb{Z}\}$, resp. $\{B(i) \mid i \in \mathbb{Z}\}$, agrees with $\mathcal{D}^b(\text{tors}(B))$, resp. $\mathcal{D}^b(\text{grproj}(B))$.*

Proof. Consult the proof of [15, Lem. 2.3]. \square

Recall that since A is connected, its Koszul dual k -algebra $A^!$ is also connected. Therefore, by first applying the functor E to (2.16) and then by using the above general Proposition 2.3 (with $B = A^!$), we obtain an induced isomorphism

$$(2.19) \quad E(\mathcal{D}_{\text{dg}}^b(\text{tors}(A))) \xrightarrow{\simeq} E(\mathcal{D}_{\text{dg}}^b(\text{grproj}(A^!))) \simeq \bigoplus_{-\infty}^{+\infty} E(k).$$

Since A is Koszul and of finite global dimension d , we have a linear free resolution

$$(2.20) \quad 0 \longrightarrow A(-d)^{\oplus \beta_d} \longrightarrow \cdots \longrightarrow A(-2)^{\oplus \beta_2} \longrightarrow A(-1)^{\oplus \beta_1} \longrightarrow A \longrightarrow k \longrightarrow 0$$

of the \mathbb{Z} -graded (right) A -module k . As mentioned in §1, the integer β_i agrees with the dimension of the k -vector space $\text{Tor}_i^A(k, k)$ (or $\text{Ext}_A^i(k, k)$).

Proposition 2.21. *Under the above isomorphisms (2.14) and (2.19), the distinguished triangle of Remark 2.13 identifies with*

$$(2.22) \quad \bigoplus_{-\infty}^{+\infty} E(k) \xrightarrow{M'} \bigoplus_{-\infty}^{+\infty} E(k) \longrightarrow E(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \xrightarrow{\partial} \bigoplus_{-\infty}^{+\infty} \Sigma E(k),$$

where M' stands for the (infinite) matrix $M'_{ij} := (-1)^j (-1)^{(i-j)} \beta'_{i-j}$.

Proof. Let $\text{NMot}(k)$ be the category of *noncommutative motives* constructed in [22, §8.2]; denoted by $\text{NMot}(k)_{\text{loc}}$ in *loc. cit.* By construction, this triangulated category comes equipped with a functor $U: \text{dgc}at(k) \rightarrow \text{NMot}(k)$ which is initial among all the functors satisfying conditions (i)-(iii). Concretely, given a functor $E: \text{dgc}at(k) \rightarrow \mathcal{T}$ satisfying conditions (i)-(iii), there exists a (unique) triangulated functor $\overline{E}: \text{NMot}(k) \rightarrow \mathcal{T}$ such that $\overline{E} \circ U \simeq E$. Moreover, \overline{E} preserves arbitrary direct sums; see [22, Thm. 8.5]. This implies that in order to prove Theorem 2.21, it suffices to show that the triangle of Remark 2.13 (with $E = U$) identifies with

$$(2.23) \quad \bigoplus_{-\infty}^{+\infty} U(k) \xrightarrow{M} \bigoplus_{-\infty}^{+\infty} U(k) \longrightarrow U(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \xrightarrow{\partial} \bigoplus_{-\infty}^{+\infty} \Sigma U(k),$$

where M stands for the (infinite) matrix $M_{ij} := (-1)^j (-1)^{(i-j)} \beta_{i-j}$. Recall from [22, §8.6] that, for every dg category \mathcal{A} , we have a natural isomorphism

$$\text{Hom}_{\text{NMot}(k)}(U(k), U(\mathcal{A})) \simeq K_0(\mathcal{A}).$$

Moreover, $U(k)$ is a compact object of the triangulated category $\text{NMot}(k)$. Therefore, since $K_0(k) \simeq \mathbb{Z}$, an endomorphism of $\oplus_{-\infty}^{+\infty} U(k)$ corresponds to an infinite matrix with integer coefficients in which every column has solely a finite number of non-zero entries. Let us denote by M the matrix corresponding to $U(\mathcal{D}_{\text{dg}}^b(\text{tors}(A))) \rightarrow U(\mathcal{D}_{\text{dg}}^b(\text{gr}(A)))$ under the isomorphisms (2.14) and (2.19) (with $E = U$). By applying the functor $\text{Hom}_{\text{NMot}(k)}(U(k), -)$ to the isomorphisms (2.14) and (2.19) (with $E = U$), we obtain induced abelian group isomorphisms

$$(2.24) \quad \oplus_{-\infty}^{+\infty} \mathbb{Z} \simeq K_0(\mathcal{D}^b(\text{grproj}(A))) \xrightarrow{\simeq} K_0(\mathcal{D}^b(\text{gr}(A)))$$

$$(2.25) \quad K_0(\mathcal{D}^b(\text{tors}(A))) \xrightarrow{\simeq} K_0(\mathcal{D}^b(\text{grproj}(A^1))) \simeq \oplus_{-\infty}^{+\infty} \mathbb{Z}.$$

The element $1 \in \mathbb{Z}$, placed at the j^{th} component of the direct sum $\oplus_{-\infty}^{+\infty} \mathbb{Z}$, corresponds under (2.25) to the Grothendieck class $[\Sigma^{-j}k(-j)] = (-1)^j[k(-j)] \in K_0(\mathcal{D}^b(\text{tors}(A)))$. In the same vein, the element $1 \in \mathbb{Z}$, placed at the i^{th} component of the direct sum $\oplus_{-\infty}^{+\infty} \mathbb{Z}$, corresponds under (2.24) to the Grothendieck class $[A(-i)] \in K_0(\mathcal{D}^b(\text{gr}(A)))$. Thanks to the above linear free resolution (2.20), we have moreover the following equality $[k(-j)] = \sum_{i=0}^d (-1)\beta_i[A(-i-j)]$ in the Grothendieck group $K_0(\mathcal{D}^b(\text{gr}(A)))$. The above considerations allow us to conclude that the $(i, j)^{\text{th}}$ entry of the matrix M is given by the integer $(-1)^j(-1)^{(i-j)}\beta_{i-j}$. This finishes the proof. \square

We now have all the ingredients necessary for the conclusion of the proof of Theorem 1.2(i). Let $o \in \mathcal{T}$ be a compact object. By applying the functor $\text{Hom}_{\mathcal{T}}(o, -)$ to the triangle (2.22), we obtain an induced long exact sequence of R -modules:

$$\cdots \rightarrow \oplus_{-\infty}^{+\infty} E_m^o(k) \xrightarrow{M'} \oplus_{-\infty}^{+\infty} E_m^o(k) \rightarrow E_m^o(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \xrightarrow{\partial} \oplus_{-\infty}^{+\infty} E_{m-1}^o(k) \rightarrow \cdots$$

Since $M'_{ij} = (-1)^j(-1)^{(i-j)}\beta'_{i-j}$, with $\beta'_0 = 1$ and $\beta'_r = 0$ whenever $r \notin \{0, \dots, d'\}$, a simple matrix computation shows that the preceding homomorphism M' of R -modules is injective. Consequently, the long exact sequence breaks-up into short exact sequences of R -modules:

$$(2.26) \quad 0 \longrightarrow \oplus_{-\infty}^{+\infty} E_m^o(k) \xrightarrow{M'} \oplus_{-\infty}^{+\infty} E_m^o(k) \longrightarrow E_m^o(\text{qgr}(A)) \longrightarrow 0.$$

Thanks to Lemma 2.28 below and to the definition of the homomorphism ϕ (see below), we also have the following short exact sequences of R -modules:

$$0 \rightarrow R[t, t^{-1}] \otimes_R E_m^o(k) \xrightarrow{\phi \otimes \text{id}} R[t, t^{-1}] \otimes_R E_m^o(k) \rightarrow R[t]/\langle h'_A(t)^{-1} \rangle \otimes_R E_m^o(k) \rightarrow 0.$$

Now, consider the Poincaré polynomial $p_A(t) := \sum_{i=0}^d (-1)^i \beta_i t^i$ (and $p'_A(t) := \sum_{i=0}^{d'} (-1)^i \beta'_i t^i$). Thanks to the linear free resolution (2.20), we have $h_A(t)^{-1} = p_A(t)$ (and $h'_A(t)^{-1} = p'_A(t)$). This implies that under the canonical isomorphism between $\oplus_{-\infty}^{+\infty} E_m^o(k)$ and $R[t, t^{-1}] \otimes_R E_m^o(k)$, the matrix M' corresponds to the homomorphism $\phi \otimes \text{id}$. Consequently, we obtain induced R -module isomorphisms

$$(2.27) \quad E_m^o(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \xrightarrow{\simeq} R[t]/\langle h'_A(t)^{-1} \rangle \otimes_R E_m^o(k) \quad m \in \mathbb{Z}.$$

This concludes the proof of Theorem 1.2(i).

Lemma 2.28. *We have the following short exact sequence of R -modules*

$$0 \longrightarrow R[t, t^{-1}] \xrightarrow{\phi} R[t, t^{-1}] \longrightarrow R[t]/\langle h'_A(t)^{-1} \rangle \longrightarrow 0,$$

where ϕ stands for the homomorphism $p(t) \mapsto p(-t) \cdot h'_A(t)^{-1}$.

Proof. Since $h'_A(0)^{-1} = 1$, the homomorphism ϕ is injective. Moreover, we have the following natural isomorphisms

$$\text{cokernel}(\phi) = R[t, t^{-1}]/\text{Im}(\phi) \stackrel{(a)}{\simeq} R[t, t^{-1}]/\langle h'_A(t)^{-1} \rangle \stackrel{(b)}{\simeq} R[t]/\langle h'_A(t)^{-1} \rangle,$$

where (a) follows from the fact that the homomorphisms ϕ and $-\cdot h'_A(t)^{-1}$ have the same image, and (b) from the fact that the polynomial t is invertible in $R[t]/\langle h'_A(t)^{-1} \rangle$ (this follows from the fact that $h'_A(0)^{-1} = 1$). This concludes the proof. \square

We now have all the ingredients necessary for the conclusion of the proof of Theorem 1.2(ii). Consider the following composition

$$(2.29) \quad \bigoplus_{n=0}^{d'-1} E(k) \longrightarrow \bigoplus_{-\infty}^{+\infty} E(k) \longrightarrow E(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))).$$

By assumption, the triangulated category \mathcal{T} is compactly generated. Therefore, the morphism (2.29) is invertible if and only if for every compact object $o \in \mathcal{T}$ the induced R -module homomorphisms

$$(2.30) \quad \bigoplus_{n=0}^{d'-1} E_m^o(k) \longrightarrow E_m^o(\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))) \quad m \in \mathbb{Z}$$

are invertible. Under the canonical identification $\bigoplus_{n=0}^{d'-1} R \otimes_R E_m^o(k) \simeq \bigoplus_{n=0}^{d'-1} E_m^o(k)$, the composition of (2.30) with (2.27) corresponds to the R -module homomorphisms:

$$((1, t, \dots, t^{d'-1}): \bigoplus_{n=0}^{d'-1} R \longrightarrow R[t]/\langle h'_A(t)^{-1} \rangle) \otimes_R E_m^o(k) \quad m \in \mathbb{Z}.$$

By assumption, we have $1/\beta' \in R$. Therefore, the factorization algorithm for polynomials applied to $R[t]$ allows us to conclude that the R -module homomorphism $(1, t, \dots, t^{d'-1})$ is invertible. This implies that the induced R -module homomorphisms (2.30) are also invertible, and so the proof of Theorem 1.2(ii) is finished.

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